1901001101020001
EXAMINATION FEBRUARY-MARCH 2024
MASTER OF ARTS PART-I
MATHEMATICS
COMPLEX ANALYSIS - LEVEL 2

[**Time: As Per Schedule]** **[Max. Marks:100]**

Instructions:
1. Fill up strictly the following details on your answer book
a. Name of the Examination: **MASTER OF ARTS PART-1**
b. Name of the Subject: **MATHEMATICS COMPLEX ANALYSIS – LEVEL 2**
c. Subject Code No: **1901001101020001**
2. Sketch neat and labelled diagram wherever necessary.
3. Figures to the right indicate full marks of the question.
4. All questions are compulsory.
5. There are five questions in this question paper.

Seat No:

Student's Signature

Q.1

A. Let z_1, z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and $|z_2|$ has negative real part than prove that $\frac{z_1+z_2}{z_1-z_2}$ has purely imaginary expression. **7**

B. State and prove De Moivre's Theorem. **7**

C. Show that **6**
1. $|z_1 + z_2| \leq |z_1| + |z_2|$
2. $\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$
3. $|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = 1 + |\bar{z}_1|^2|z_2|^2 - (|z_1|^2 + |z_2|^2)$

OR

A. Prove that the modulus of the product of two complex numbers is the product of their moduli. **7**

B. Express the following complex numbers in the form $x + iy$, where x, y are real numbers. **7**
(a) $(-1+3i)^{-1}$ (b) $(1 + i)i(2 - i)$ (c) $(7+\pi i)(\pi + i)$

C. The modulus of the difference of two complex numbers is greater than or equal to the difference of their moduli. **6**

Q.2 A. If f and g be functions defined on the open set U. If f and g are differentiable at z then prove that fg and f/g are also differentiable on U. **7**

B. If $w = f(z)$ and f is differentiable at z, and g is differentiable at w then prove that (g o f) is differentiable at z, and (g o f)'(z) = g'(f(z))f'(z). **7**

C. State and Prove Cauchy- Riemann theorem. **6**

OR

A. If $f(z) = u + iv$ be an analytic function then show that u and v are harmonic function. **7**

B. If f is analytic in domain D then **7**
1. If $f'(z) = 0$ in D, f is constant.
2. If one of $|f|$, Re(f), Im(f) is constant in D, f is constant.

C. Describe the Method to find an analytic function f whose real part is a given harmonic function $u(x, y)$. **6**

Q.3 A. Prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|u|^n = n(n-1)|u|^{n-2}|f'(z)|^2$ **7**

B. Evaluate: $I = \int_\gamma f(z)dz; f(Z) = z^2$ Where γ is along the horizontal segment 0 to 1 and then by vertical segment from 1 to 1+2i. **7**

C. Evaluate:
1. $I = \int_\gamma y\, dz$, where γ is the circle |z|=R. **6**
2. $I = \int_\gamma \overline{z}\, dz$ where γ circumference of the square [0, 1] x [0, 1].

OR

A. State and prove Cauchy - Goursat Theorem **7**

B. Let φ be a complex valued function which is continuous in an open set D containing an arc γ. Then $\forall z \notin \gamma$ the function F_n is defined by $F(z) = \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)} d\zeta; n = 1,2,3, \ldots$ is analytic and satisfies the equation **7**

$F'_n(z) = nF_{n+1}(z)$ or equivalently
$F^k(z) = k! \int_\gamma \frac{\phi(\zeta)}{(\zeta - z)^{k+1}} d\zeta; k = 1,2,3, \dots$ with $F_1(z) = F(z)$

C. State and prove Morera's Theorem **6**

Q.4 A. Evaluate following integral by the method of calculus of residue **7**

1. $\int_c \frac{e^z}{z(z-1)} dz$
2. $\int_c \frac{1}{(z-1)(z-2)} dz$

B. Find residue of Ø(z) = cot z at the points $z_n = n\pi \; for \; n = 1,2, \dots.$ **7**
What is the nature of singularity at $z = \infty$? Justify your answer.

C. Discuss the nature of singularities of the following functions **6**

1. $f(z) = \tan z$
2. $f(z) = \frac{1}{z(1-z^2)}$
3. $f(z) = \frac{z}{1+z^4}$

OR

A. If $f(z) and\; g(z)$ are analytic in Ω and if $f(z) = g(z)$ on a set which has a limit point in Ω, then prove that $f(z)$ is identically equal to $g(z)$. **7**

B. State and prove Cauchy's residue theorem. **7**

C. Prove that $\lim_{z \to \infty} -zf(z) = Res(z = \infty)$provided $f(z)$ is analytic at $z = \infty$. **6**

Q.5 A. Prove that $\int_0^{2\pi} e^{-\cos\theta} \cos(n\theta + sin\,\theta)\, d\theta = \frac{2\pi(-1)^n}{n!}$ **20**

B. Prove that $\int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2}$

C. Prove that $\int_0^\infty \frac{x - \sin x}{(a^2+x^2)x^3} dx = \frac{\pi}{2a^4}\left[\frac{1}{2}a^2 - a + 1 - e^{-a}\right], a > 0$

D. Prove that $\int_0^\infty \frac{x^{1/6} \log x}{(1+x)^2} = 2\pi - \frac{\pi^2}{\sqrt{3}}$

OR

20

A. Prove that $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \frac{\pi}{6}$

B. Evaluate the integral $\int_{-\infty}^{\infty} \frac{1}{(x^2+4)(x^2+9)^2} dx,$

C. Evaluate $\int_0^{\infty} \frac{x^{a-1}}{1-x} dx \,; 0 < a < 1$

D. Prove that if $-1 < a < 3, then\ \int_0^{\infty} \frac{x^a}{(1+x^2)^2} dx = \frac{\pi}{4}(1-a)\sec\left(\frac{\pi a}{2}\right)$

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